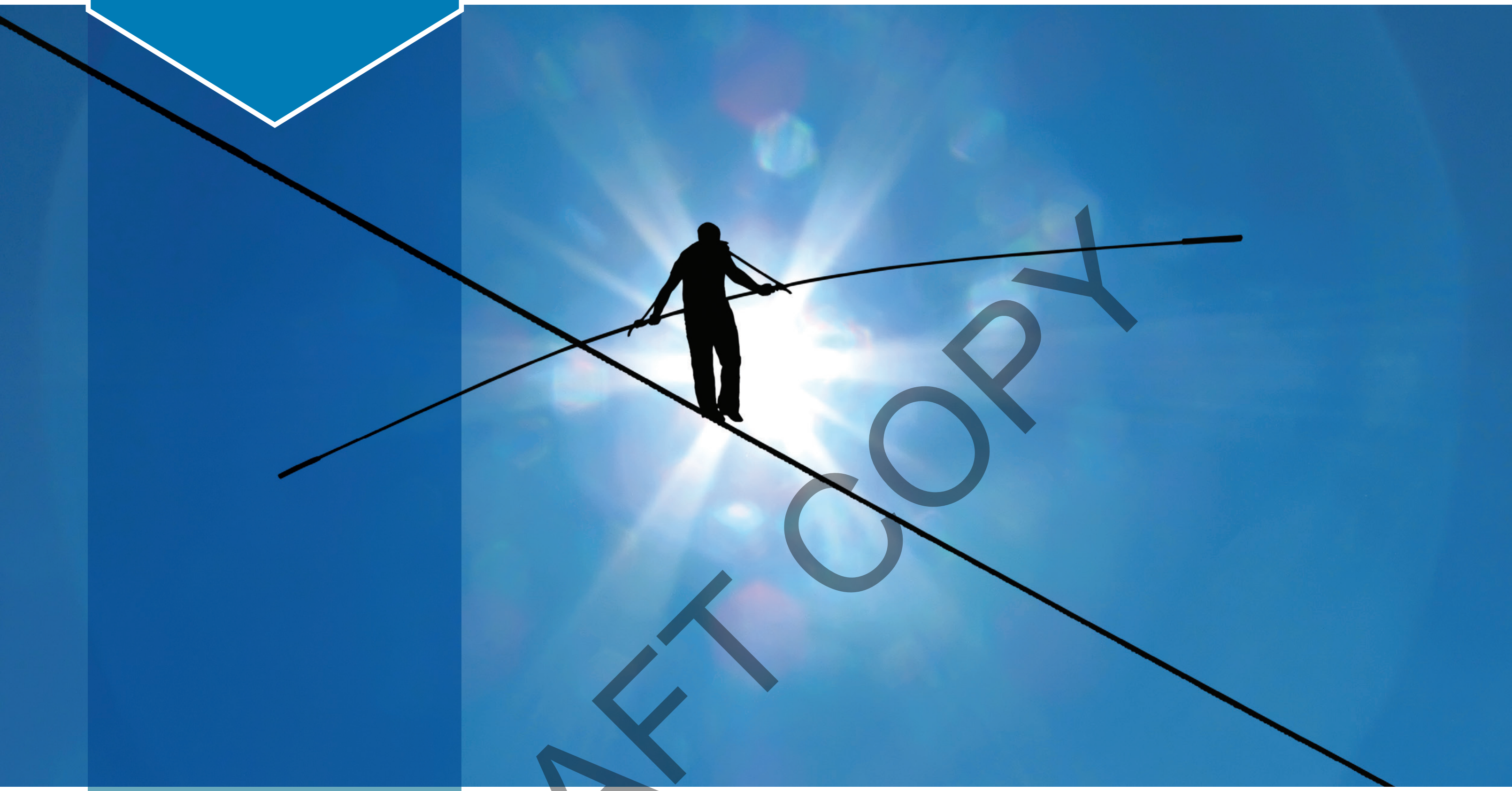
2 Forces and motion



> *In the beginning (if there was such a thing) God created Newton's laws of motion together with the necessary masses and forces. This is all; everything beyond this follows from the development of appropriate mathematical methods by means of deduction.*
>
> Albert Einstein

→ This tightrope walker is stationary. There are a number of forces acting on the walker which cancel each other out, resulting in no motion. In order for that to be possible, the cable must make small angles to the horizontal so that the vertical components of the tension can cancel out the weight of the walker. In that case, the tensions in the cable will be greater than the walker's weight.

1 Forces and Newton's laws of motion

Modelling vocabulary

Mechanics is about modelling the real world. In order to do this, suitable simplifying assumptions are often made so that mathematics can be applied to situations and problems. This process involves identifying factors that can be neglected without losing too much accuracy. Here are some commonly used modelling terms which are used to describe such assumptions.

- negligible: small enough to ignore
- inextensible: for a string with negligible stretch
- light: for an object with negligible mass
- particle: an object with negligible dimensions
- smooth: for a surface with negligible friction
- uniform: the same throughout.

Forces

A force is defined as the physical quantity that causes a change in motion. As it depends on magnitude and direction, it is a vector quantity.

Forces can start motion, stop motion, speed up or slow down objects, or change the direction of their motion. In real situations, several forces usually act on an object. The sum of these forces, known as the resultant force, determines whether or not there is a change of motion.

There are several types of force that you often use.

Note

g varies around the world, with $9.8\,\mathrm{m\,s^{-2}}$ being a typical value. Singapore, at 9.766, has one of the lowest values, and Helsinki, with 9.825, has one of the highest.

The force of gravity

Every object on or near the Earth's surface is pulled vertically downwards by the force of gravity. The size of the force on an object of mass M kg is Mg newtons where g is a constant whose value is about $9.8\,\mathrm{m\,s^{-2}}$. The force of gravity is also known as the **weight** of the object.

Tension and thrust

When a string is pulled, as in Figure 2.1, it exerts a **tension** force opposite to the pull. The tension acts along the string and is the same throughout the string. A rigid rod can exert a tension force in a similar way to a string when it is used to support or pull an object. It can also exert a **thrust** force when it is in compression, as in Figure 2.2. The thrust acts along the rod and is the same throughout the rod.

The tension on either side of a smooth pulley is the same, as shown in Figure 2.3.

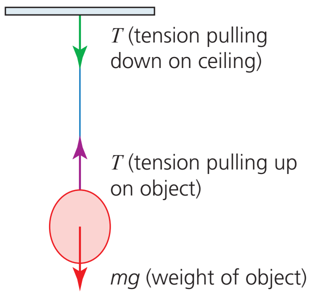


Figure 2.1

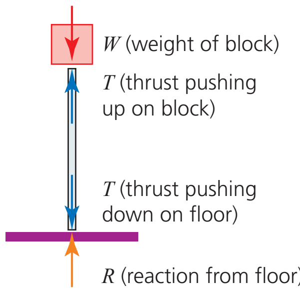


Figure 2.2

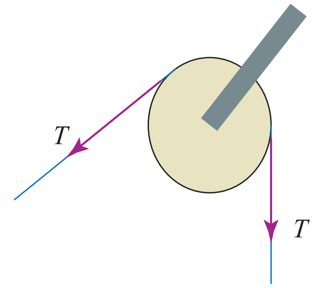


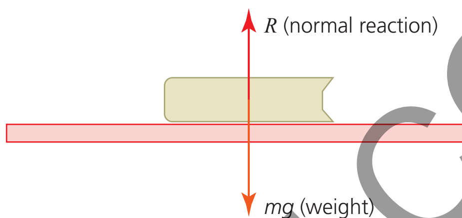
Figure 2.3

Normal reaction

A book resting on a table is subjected to two forces, its weight and the **normal reaction** of the table. It is called normal because its line of action is normal (at right angles) to the surface of the table. Since the book is in equilibrium, the normal reaction is equal and opposite to the weight of the book; it is a positive force.

Note

In Figure 2.4, the normal reaction is vertical but this is not always the case. For example, the normal reaction on an object on a slope is perpendicular to the slope.



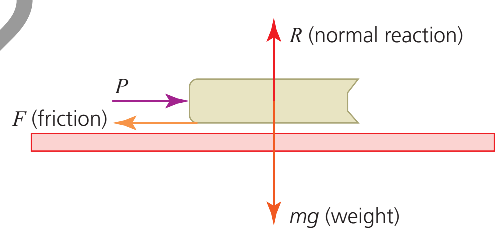
The normal reaction force is also often called the normal contact force.

Figure 2.4

If the book is about to lose contact with the table (which might happen, for instance, if the table is accelerating rapidly downwards), the normal force becomes zero.

Frictional force

In this diagram, the book on the table is being pushed by a force P parallel to the surface. The book remains at rest because P is balanced by a **frictional force**, F, in the opposite direction to P. The magnitude of the frictional force is equal to the pushing force, i.e. $P = F$



The frictional force is also sometimes called the frictional contact force.

Figure 2.5

If P is increased and the book starts to move, F is still present but now $P > F$. Friction always acts in the opposite direction to the motion. Friction may prevent the motion of an object or slow it down if it is moving.

Driving force

In problems about moving objects such as cars, all the forces acting along the line of motion can usually be reduced to two or three: the **driving force**, the **resistance** to motion and, possibly, a **braking force**.

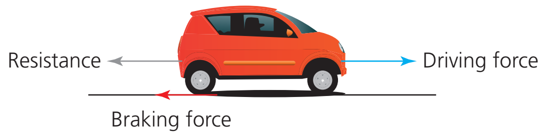


Figure 2.6

Example 2.1

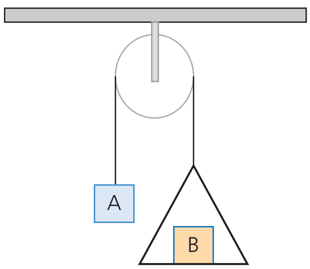


Figure 2.7

Figure 2.7 shows a block A of mass 10 kg connected to a light scale pan by a light inextensible string that passes over a light smooth pulley. The scale pan holds block B, also of mass 10 kg. The system is in equilibrium.

(i) On separate diagrams, show all the forces acting on each of the masses, the scale pan and the pulley.

(ii) Find the value of the tension in the string.

(iii) Find the tension in the rod holding the pulley.

(iv) Find the normal reaction of B on the scale pan.

Solution

(i)

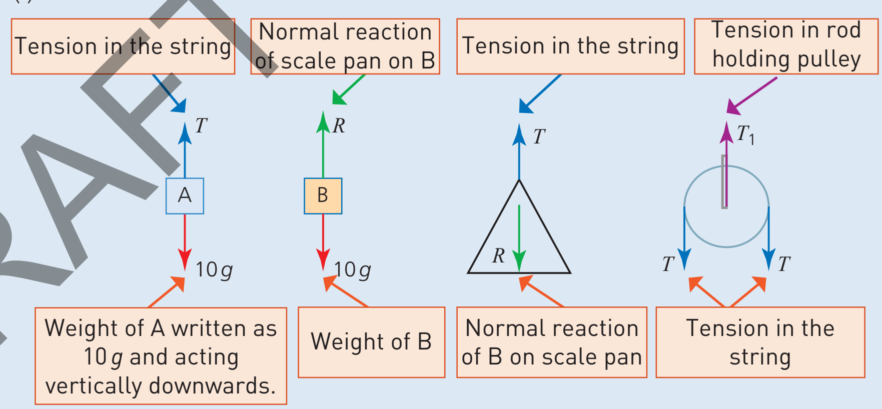


Figure 2.8

(ii) Block A is in equilibrium: $\Rightarrow$ $T = 10g = 98$ N.

(iii) The pulley is in equilibrium: $\Rightarrow$ $T_1 = 2T = 196$ N.

(iv) Block B is in equilibrium: $\Rightarrow$ $R = 10g = 98$ N.

Newton’s laws of motion

1 Every object continues in a state of rest or uniform motion in a straight line unless it is acted on by a resultant external force.

2 The acceleration of an object is proportional to, and in the same direction as, the resultant of the forces acting on the object.

$\mathbf{F} = m\mathbf{a}$

$\mathbf{F}$ is the resultant force.
m is the mass of the object.
$\mathbf{a}$ is the acceleration.

Historical note

Isaac Newton was born in Lincolnshire in 1642. He was not an outstanding scholar either as a schoolboy or as a university student, yet later in life he made remarkable contributions in dynamics, optics, astronomy, chemistry, music theory and theology. He became Member of Parliament for Cambridge University and later Warden of the Royal Mint. His tomb in Westminster Abbey reads 'Let mortals rejoice that there existed such and so great an ornament to the Human Race'.

Notice that this is a vector equation, since both the magnitudes and directions of the resultant force and the acceleration are involved. If the motion is along a straight line it is often written in scalar form as $F = ma$.

3 When one object exerts a force on another there is always a reaction, which is equal and opposite in direction to the acting force.

Equation of motion

The equation resulting from Newton's second law is often described as an **equation of motion**, as in the following examples.

Example 2.2

An empty bottle of mass 0.5 kg is released from a submarine and rises with an acceleration of $0.75\,\mathrm{m\,s^{-2}}$. The water causes a resistance of 1.1 N.

(i) Draw a diagram showing the forces acting on the bottle and the direction of its acceleration.

(ii) Write down the equation of motion of the bottle.

(iii) Find the size of the buoyancy force.

Solution

(i) The forces acting on the bottle and the acceleration are shown in this diagram.

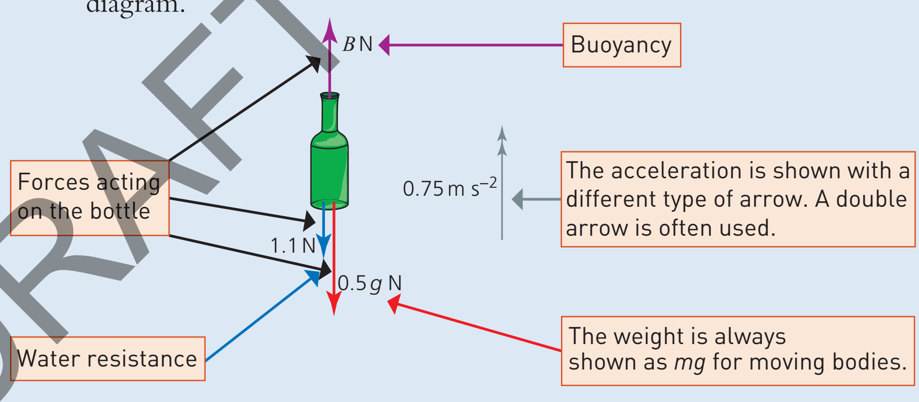


Figure 2.9

(ii) The resultant force acting on the bottle is $(B - 0.5g - 1.1)$ upwards.

The resulting equation

$$B - 0.5g - 1.1 = 0.5a$$

is called the **equation of motion**.

(iii) $B - 4.9 - 1.1 = 0.375$ ← 0.5 × 0.75

$$B = 6.375$$

The buoyancy force on the bottle is 6.375 N.

Example 2.3

A car of mass 900 kg travels at a constant speed of $20\,\text{m s}^{-1}$ along a straight horizontal road. Its engine is producing a driving force of 500 N.

(i) What is the resistance to its motion?

Later the driving force is removed and the car is brought to rest in a time of 5 s with the same resistance to motion.

(ii) Find the force created by the brakes, assuming it to be constant.

Solution

(i) The car is travelling at constant speed, so that the resultant force acting on the car is zero.

Resultant = $500 - R$

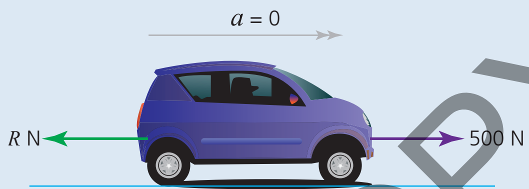


Figure 2.10

Let the resistive force be R N.

$$500 - R = 0$$
$$R = 500$$

The resistive force is 500 N.

(ii) The car is slowing down.

So you expect a to be negative.

Resultant = $-B - 500$

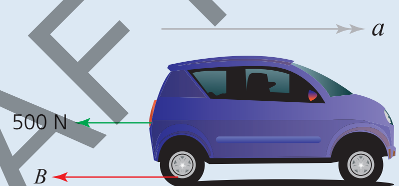


Figure 2.11

B is constant, so that a is also constant and you can use the constant acceleration formulae.

The equation of motion is $-B - 500 = 900a$ ①

$0 = 20 + a \times 5$

Use $v = u + at$ with $u = 20$, $v = 0$ and $t = 5$.

$a = -4\,\text{m s}^{-2}$

Substituting in ① $-B - 500 = 900 \times -4$

$\Rightarrow B = 3100\,\text{N}$

The braking force is 3100 N.

Example 2.4

Two boxes A and B are descending vertically supported by a parachute. Box A has mass 100 kg. Box B has mass 75 kg and is suspended from box A by a light vertical wire. Both boxes are descending with downwards acceleration $2\,\mathrm{m\,s^{-2}}$.

(i) Draw a labelled diagram showing all the forces acting on box A and another diagram showing all the forces acting on box B.

(ii) Write down separate equations of motion for box A and for box B.

(iii) Find the tensions in both wires.

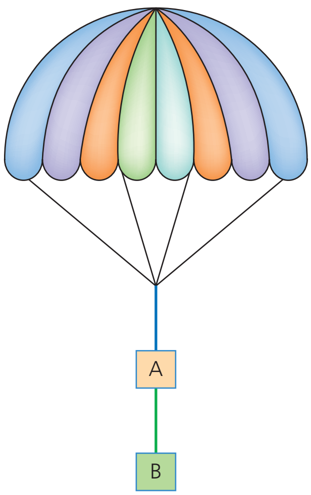


Figure 2.12

Solution

(i)

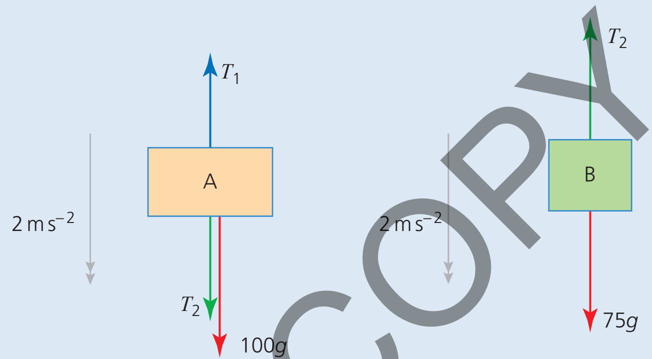


Figure 2.13

(ii) Box A: The resultant downwards force is $T_2 + 100g - T_1$ so the equation of motion is

$T_2 + 100g - T_1 = 200$ ① ← Since $100a = 100 \times 2 = 200$

Box B: The resultant downwards force is $75g - T_2$ so the equation of motion is

$75g - T_2 = 150$ ② ← Since $75a = 75 \times 2 = 150$

(iii) From ②: $T_2 = 75 \times 9.8 - 150$

$= 585\,\mathrm{N}$

Substituting in ①: $T_1 = 585 + 100 \times 9.8 - 200$

$= 1365\,\mathrm{N}$

The tension in the blue wire linking A to the parachute is 1365 N. The tension in the green wire linking A to B is 585 N.

Note

T_1 is the tension in the blue wire linking A to the parachute. T_2 is the tension in the green wire linking A to B. $T_1 \neq T_2$

Exercise 2.1

① Find the acceleration produced when a force of 100 N acts on an object

(i) of mass 15 kg

(ii) of mass 10 g

(iii) of mass 1 tonne.

② A bullet of mass 20 g is fired into a wall with a velocity of $400\,\mathrm{m\,s^{-1}}$. The bullet penetrates the wall to a depth of 10 cm. Find the resistance of the wall, assuming it to be uniform.

③ A car of mass 1200 kg is travelling along a straight level road.

(i) Calculate the acceleration of the car when a resultant force of 2400 N acts on it in the direction of its motion. How long does it take the car to increase its speed from $4\,\text{m}\,\text{s}^{-1}$ to $12\,\text{m}\,\text{s}^{-1}$?

The car has an acceleration of $1.2\,\text{m}\,\text{s}^{-2}$ when there is a driving force of 2400 N.

(ii) Find the resistance to motion of the car.

④ A load of mass 5 kg is held on the end of a string. Calculate the tension in the string when

(i) the load is raised with an acceleration of $2.5\,\text{m}\,\text{s}^{-2}$

(ii) the load is lowered with an acceleration of $2.5\,\text{m}\,\text{s}^{-2}$

(iii) the load is raised with a constant speed of $2\,\text{m}\,\text{s}^{-1}$

(iv) the load is raised with a deceleration of $2.5\,\text{m}\,\text{s}^{-2}$.

⑤ A block A of mass 10 kg is connected to a block B of mass 5 kg by a light inextensible string passing over a smooth fixed pulley. The blocks are released from rest with A 0.3 m above ground level, as shown in Figure 2.14.

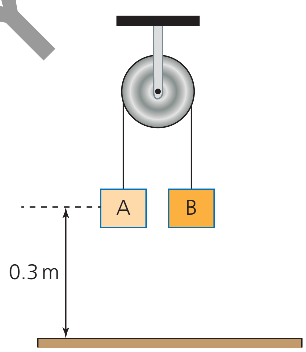


Figure 2.14

(i) Find the acceleration of the system and the tension in the string.

(ii) Find the speed of the masses when A hits the ground.

(iii) How far does B rise after A hits the floor and the string becomes slack?

⑥ A block A of mass 10 kg is lying on a smooth horizontal table. Light inextensible strings connect A to particle B of mass 6 kg and particle C of mass 4 kg, which hang freely over smooth pulleys at the edge of the table.

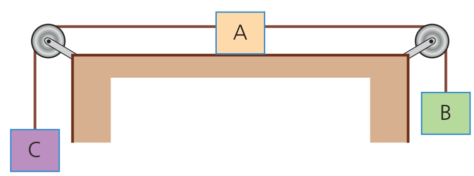


Figure 2.15

(i) Draw force diagrams to show the forces acting on each mass.

(ii) Write down separate equations of motion for A, B and C.

(iii) Find the acceleration of the system and the tensions in the strings.

⑦ A truck of mass 1250 kg is towing a trailer of mass 350 kg along a horizontal straight road. The engine of the truck produces a driving force of 2500 N. The truck is subjected to a resistance of 250 N and the trailer to a resistance of 300 N.

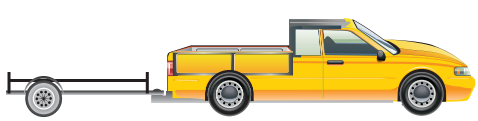


Figure 2.16

(i) Show, in separate diagrams, the horizontal forces acting on the truck and the trailer.

(ii) Find the acceleration of the truck and trailer.

(iii) Find the tension in the coupling between the truck and the trailer.

⑧ A train consists of a locomotive and five trucks with masses and resistances to motion as shown in Figure 2.17. The engine provides a driving force of 29 000 N. All the couplings are light, rigid and horizontal.

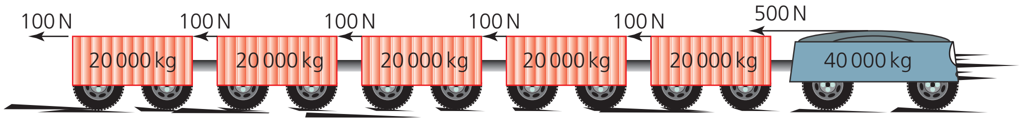


Figure 2.17

(i) Show that the acceleration of the train is $0.2\,\mathrm{m\,s^{-2}}$.

(ii) Find the force in the coupling between the last two trucks.

With the driving force removed, brakes are applied, so adding additional resistances of 3000 N to the locomotive and 2000 N to each truck.

(iii) Find the new acceleration of the train.

(iv) Find the force in the coupling between the last two trucks.

⑨ Block A of mass 2 kg is connected to a light scale pan by a light inextensible string which passes over a smooth fixed pulley.

The scale pan holds two blocks, B and C, of masses 0.5 and 1 kg, as shown in Figure 2.18.

(i) Draw diagrams showing all the forces acting on each of the three particles.

(ii) Write down equations of motion for each of A, B and C.

(iii) Find the acceleration of the system, the tension in the string, the reaction force between B and C and the reaction between C and the scale pan.

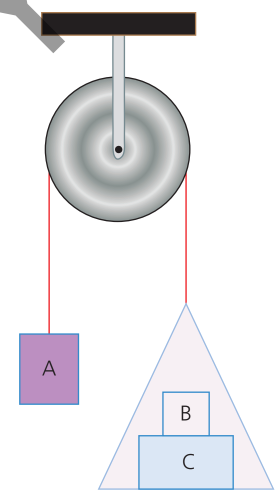


Figure 2.18

⑩ A block A of mass 5 kg is lying on a smooth horizontal table. A light inextensible string connects A to a particle B of mass 1 kg which hangs freely over a smooth pulley at the edge of the table. B is connected to a third particle C of mass 2 kg by another string, as shown in Figure 2.19.

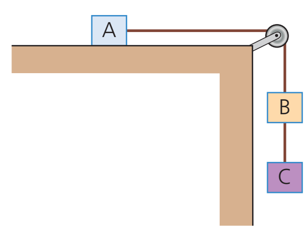


Figure 2.19

(i) Draw diagrams showing all the forces acting on each of the three blocks.

(ii) Write equations of motion for each of A, B and C.

(iii) Find the acceleration of the system and the tension in each string.

2 Working with vectors

A force is a physical quantity that causes a change of motion. A force can start the motion of an object, stop its motion, make it move faster or slower, or change the direction of its motion. By its very nature, a force is a vector quantity just like displacement, velocity or acceleration. It has magnitude and direction, unlike scalar quantities such as distance, speed, mass or time, which have magnitude only.

Notation and representation

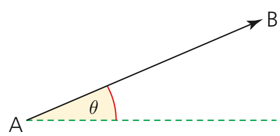


Figure 2.20

A vector can be represented by a directed line segment in a diagram.

In writing, $\overrightarrow{AB}$ represents the vector with magnitude AB. The magnitude of the vector can be written as $|\overrightarrow{AB}|$. The direction is given by the angle θ which AB makes with a fixed direction, often the horizontal.

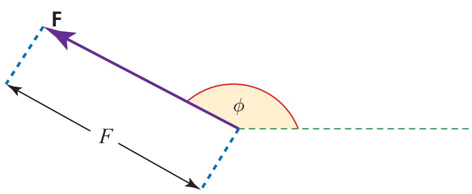


Figure 2.21

In Figure 2.21, **F** is a vector with magnitude $F = |\mathbf{F}|$ and direction ϕ.

Vectors are often written using lowercase letters, like **a**, **b** and **c**. It is common to use **a** for the vector $\overrightarrow{OA}$ where O is the origin.

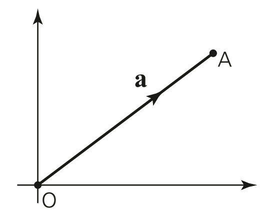


Figure 2.22

To determine the direction of a vector in the xy plane, a mathematical convention is used. Starting from the x-axis, angles measured anticlockwise are positive and angles in a clockwise direction are negative.

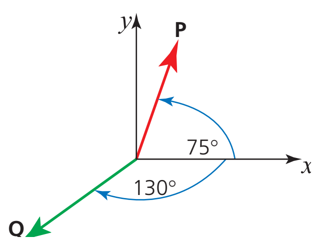


Figure 2.23

In the example shown in Figure 2.23, **P** has direction +75° and **Q** has direction −130°.

Adding vectors

One way to add vectors is to draw them one after another, i.e. where one finishes the next one starts (Figure 2.24).

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

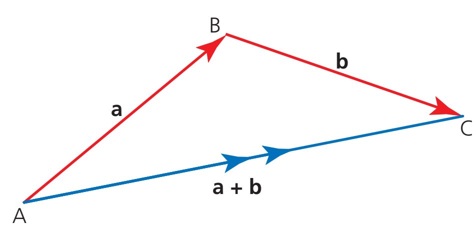


Figure 2.24

Alternatively, you can make **a** and **b** start at the same place and take the diagonal of the ensuing parallelogram (Figure 2.25).

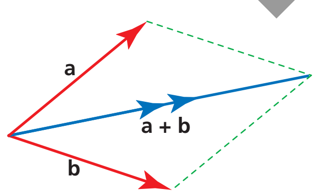


Figure 2.25

This gives the same result, because opposite sides of a parallelogram are equal and in the same direction, so that **b** is repeated at the top right of the parallelogram.

Resultant forces

The resultant of a number of forces is equal to the sum of these forces. As each force is a vector, the resultant is a vector starting at the start point of the first force and ending at the end point of the last force.

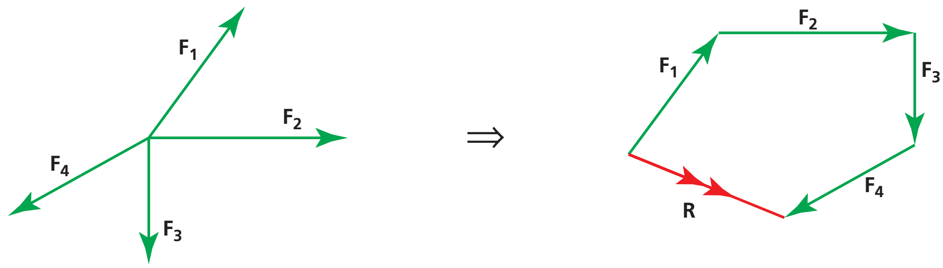


Figure 2.26

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$$

The resultant **R** of the four forces ($\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$) can be found by drawing consecutive lines representing the vectors. The line which completes the polygon is the resultant.

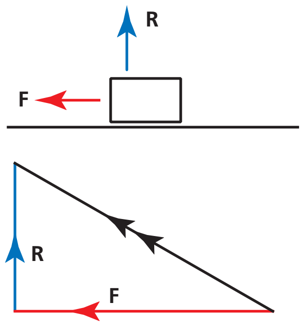


Figure 2.27

You have met the contact forces of normal reaction force **R** and frictional force **F**. The resultant of them is the total contact force with magnitude $\sqrt{F^2+R^2}$.

Components of a vector

Finding components is the reverse process of adding two vectors. It involves splitting a vector into two perpendicular components.

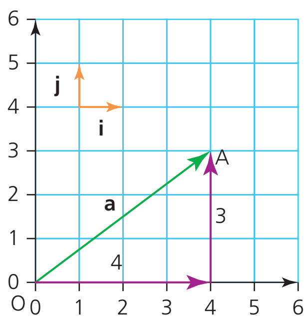


Figure 2.28

The result is often described using unit vectors **i** and **j** along the x and y axes, respectively. The vector **a** in Figure 2.28 may be written as $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$.

Alternatively, **a** can be written as the *column vector* $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

In general, if **i** and **j** are unit vectors along the x and y direction, respectively, then **a** can be written in terms of components as

$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$ or in column vector form

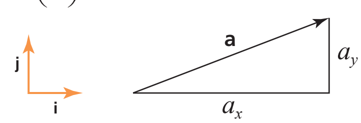


Figure 2.29

$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ since

$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$

Example 2.5

The four vectors **a**, **b**, **c** and **d** are shown in the diagram.

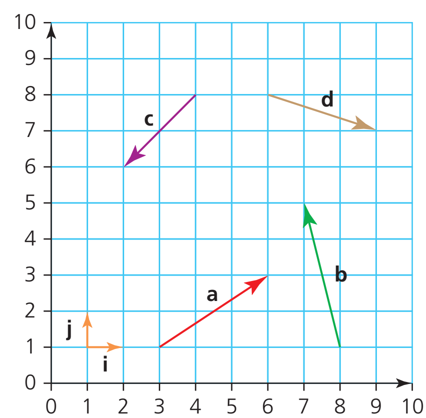


Figure 2.30

(i) Write them in component form and as column vectors.

(ii) Draw a diagram to show the vectors $2\mathbf{a}$, $-\mathbf{b}$ and $2\mathbf{a} - \mathbf{b}$ and write these in both forms.

Solution

(i) $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\quad \mathbf{b} = -\mathbf{i} + 4\mathbf{j} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\quad \mathbf{c} = -2\mathbf{i} - 2\mathbf{j} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$,

$\mathbf{d} = 3\mathbf{i} - \mathbf{j} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

(ii)

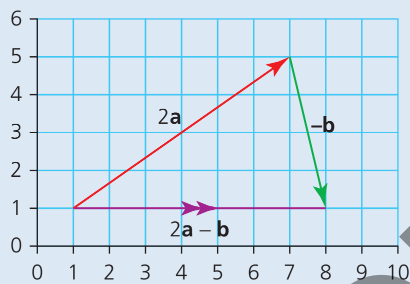


Figure 2.31

$$2\mathbf{a} = 2(3\mathbf{i} + 2\mathbf{j}) = 6\mathbf{i} + 4\mathbf{j} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$-\mathbf{b} = -(-\mathbf{i} + 4\mathbf{j}) = \mathbf{i} - 4\mathbf{j} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$2\mathbf{a} - \mathbf{b} = 2\mathbf{a} + (-\mathbf{b}) = 6\mathbf{i} + 4\mathbf{j} + \mathbf{i} - 4\mathbf{j} = 7\mathbf{i} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

Position vectors

To specify the position of an object, you define its displacement relative to a fixed origin. $\mathbf{a}$ and $\mathbf{b}$ are usually used to define the position vectors of A and B.

The vector between two points

$\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$\overrightarrow{AB} = -\mathbf{a} + \mathbf{b}$

$\quad = \mathbf{b} - \mathbf{a}$

The displacement AB can be replaced by the displacement from A to O followed by that from O to B.

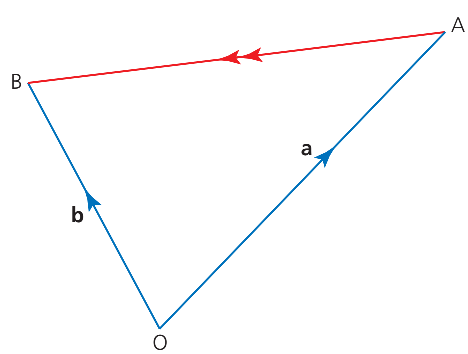


Figure 2.32

Any displacement vector $\overrightarrow{AB}$ can be written in terms of the position vectors of its two end points.

The magnitude and direction of vectors written in component form

The magnitude of a vector is just its length and can be found by using Pythagoras' theorem.

$$a = \sqrt{a_x^2 + a_y^2}$$

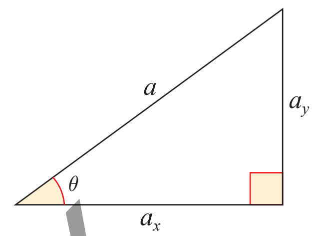


Figure 2.33

The direction is related to the angle the vector makes with the positive x-axis.

$$\theta = \arctan\left(\frac{a_y}{a_x}\right)$$

or $\theta = \arctan\left(\frac{a_x}{a_y}\right) + 180°$, depending on which quadrant θ is in.

Example 2.6

Find the magnitude and direction of each of the four vectors

$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\mathbf{c} = -2\mathbf{i} - 2\mathbf{j}$ and $\mathbf{d} = 3\mathbf{i} - \mathbf{j}$.

Solution

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Magnitude $|\mathbf{a}| = \sqrt{3^2 + 2^2} = \sqrt{13} = 3.61$

Direction $\theta = \arctan\left(\frac{2}{3}\right) = 33.7°$

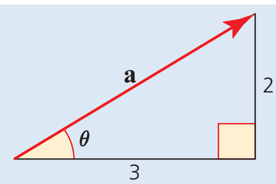


Figure 2.34

$$\mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Magnitude $|\mathbf{b}| = \sqrt{(-1)^2 + 4^2} = \sqrt{17} = 4.12$

$\tan\phi = \frac{4}{1} \Rightarrow \phi = \arctan 4 = 76.0°$

Direction $\theta = 180 - \phi = 104°$

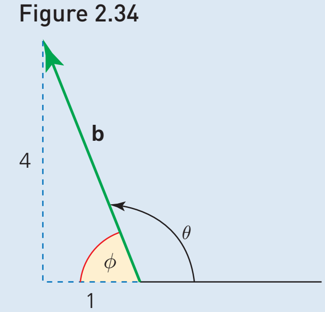


Figure 2.35

$\mathbf{c} = -2\mathbf{i} - 2\mathbf{j}$

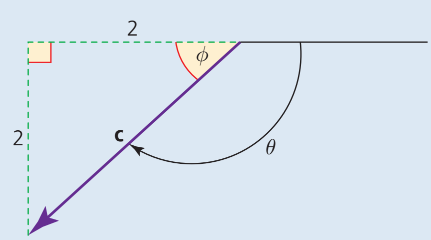


Figure 2.36

Magnitude $|\mathbf{c}| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2.83$

$\tan\phi = \frac{2}{2} = 1 \Rightarrow \phi = \arctan 1 = 45°$

Direction $\theta = -(180° - 45°) = -135°$

$\mathbf{d} = 3\mathbf{i} - \mathbf{j}$

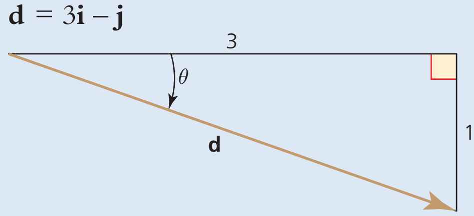


Figure 2.37

Magnitude $|\mathbf{d}| = \sqrt{3^2 + (-1)^2} = \sqrt{10} = 3.16$

Direction $\theta = -\arctan\left(\frac{1}{3}\right) = -18.4°$

Finding unit vectors along given directions

If $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$, the magnitude of $\mathbf{a}$ is $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2}$. A unit vector along $\mathbf{a}$ (denoted by $\hat{\mathbf{a}}$) has magnitude 1. The vector $\hat{\mathbf{a}}$ is parallel to $\mathbf{a}$ but has a magnitude which is scaled by the factor $\frac{1}{|\mathbf{a}|}$.

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{a_x}{\sqrt{a_x^2 + a_y^2}}\mathbf{i} + \frac{a_y}{\sqrt{a_x^2 + a_y^2}}\mathbf{j}$$

Example 2.7

(i) Find a unit vector along $\mathbf{a} = 3.5\mathbf{i} - 12\mathbf{j}$

(ii) Find a vector along $\mathbf{a}$ which is 25 units long.

Solution

(i) The magnitude of $\mathbf{a}$ is $|\mathbf{a}| = \sqrt{3.5^2 + (-12)^2} = \sqrt{12.25 + 144}$

$= \sqrt{156.25} = 12.5$

A unit vector along $\mathbf{a}$ is $\hat{\mathbf{a}} = \frac{3.5}{12.5}\mathbf{i} - \frac{12}{12.5}\mathbf{j} = 0.28\mathbf{i} - 0.96\mathbf{j}$

(ii) $\hat{\mathbf{a}}$ has magnitude 1, so you are looking for a vector along $\mathbf{a}$ which is 25 units long, i.e. $25\hat{\mathbf{a}} = 25(0.28\mathbf{i} - 0.96\mathbf{j}) = 7\mathbf{i} - 24\mathbf{j}$

Resolving a force into components in two perpendicular directions

Draw the vector $\mathbf{F}$, magnitude F, making an angle θ with the x-axis, taken as the $\mathbf{i}$ direction. Make up the right-angled triangle with $\mathbf{F}$ along the hypotenuse and the x and y components along the other two sides. These are then evaluated using trigonometry.

$$\mathbf{F} = F\cos\theta\,\mathbf{i} + F\sin\theta\,\mathbf{j} = \begin{pmatrix} F\cos\theta \\ F\sin\theta \end{pmatrix}$$

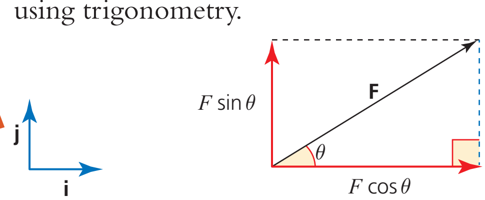


Figure 2.38

Example 2.8

Resolve a weight W N in two directions which are along and at right angles to a slope making an angle θ with the horizontal.

Solution

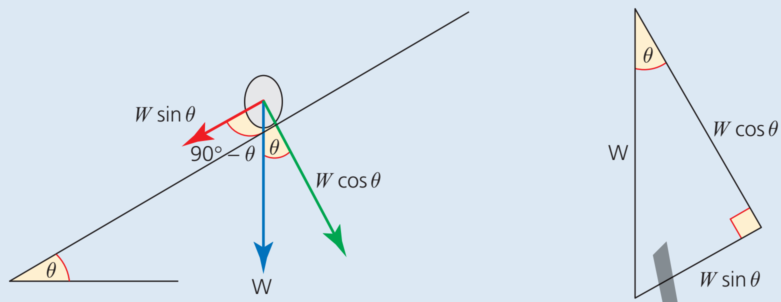


Figure 2.39

W is the hypotenuse of the right-angled triangle. The components are along the other two sides. The component parallel to the slope is $W \cos(90° - \theta) = W \sin\theta$.

The component perpendicular to the slope is $W \cos\theta$.

Example 2.9

Two forces **P** and **Q** have magnitudes 10 N and 15 N in the directions shown in the diagram.

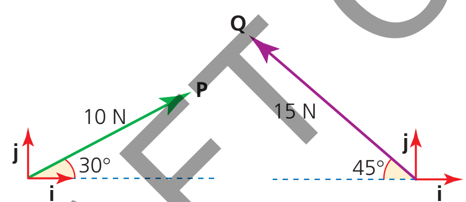


Figure 2.40

Find the magnitude and direction of the resultant force **P** + **Q**.

Solution

Note

$\mathbf{P} = 10\cos 30°\,\mathbf{i} + 10\sin 30°\,\mathbf{j}$
$= 8.66\,\mathbf{i} + 5\,\mathbf{j}$

Note

$\mathbf{Q} = -15\cos 45°\,\mathbf{i} + 15\sin 45°\,\mathbf{j}$
$= -10.61\,\mathbf{i} + 10.61\,\mathbf{j}$

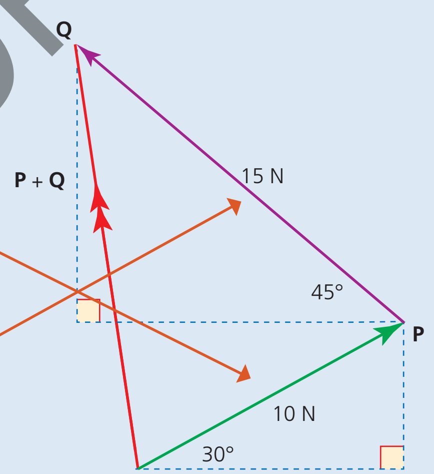


Figure 2.41

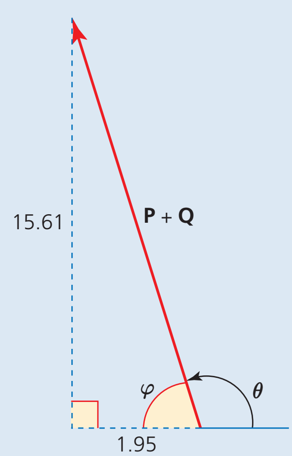


Figure 2.42

The resultant is $\mathbf{P} + \mathbf{Q} = (8.66\mathbf{i} + 5\mathbf{j}) + (-10.61\mathbf{i} + 10.61\mathbf{j})$

$= -1.95\mathbf{i} + 15.61\mathbf{j}$

It is shown in Figure 2.42.

The magnitude of the resultant is $|\mathbf{P}+\mathbf{Q}| = \sqrt{(-1.95)^2 + 15.61^2} = 15.73$

The direction of the resultant:

$$\tan\varphi = \frac{15.61}{1.95} \Rightarrow \varphi = \arctan(8.01) = 82.9°$$

$$\theta = 180° - 82.9° = 97.1°$$

The resultant force $\mathbf{P} + \mathbf{Q}$ has magnitude 15.73 and direction 97.1° relative to the positive x-axis.

Exercise 2.2

① Four vectors are given in component form by $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{b} = 6\mathbf{i} - 7\mathbf{j}$, $\mathbf{c} = -2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{d} = -5\mathbf{i} - 3\mathbf{j}$.

Find the vectors

(i) $\mathbf{a} + \mathbf{b}$ (ii) $\mathbf{b} + \mathbf{c}$ (iii) $\mathbf{c} + \mathbf{d}$

(iv) $\mathbf{a} + \mathbf{b} + \mathbf{d}$ (v) $\mathbf{a} - \mathbf{b}$ (vi) $\mathbf{d} - \mathbf{b} + \mathbf{c}$

② A, B, C are the points (1, 2), (5, 1) and (7, 8).

(i) Write down in terms of **i** and **j** the position vectors of these three points.

(ii) Find the component form of the displacements $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

(iii) Draw a diagram to show the position vectors of A, B and C and your answers to part (ii).

③ Three vectors **a**, **b** and **c** are given by $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

R is the endpoint of the displacement $2\mathbf{a} - 3\mathbf{b} + \mathbf{c}$ and (1, –2) is the starting point. What is the position vector of R?

④ Find the magnitude and direction of the following vectors

(i) $12\mathbf{i} - 5\mathbf{j}$ (ii) $7\mathbf{i} + 24\mathbf{j}$ (iii) $-\mathbf{i} + \mathbf{j}$

(iv) $3\mathbf{i} + 4\mathbf{j}$ (v) $2\mathbf{i} - 3\mathbf{j}$ (vi) $-\mathbf{i} - 2\mathbf{j}$

⑤ Write down the following vectors in component form in terms of **i** and **j**

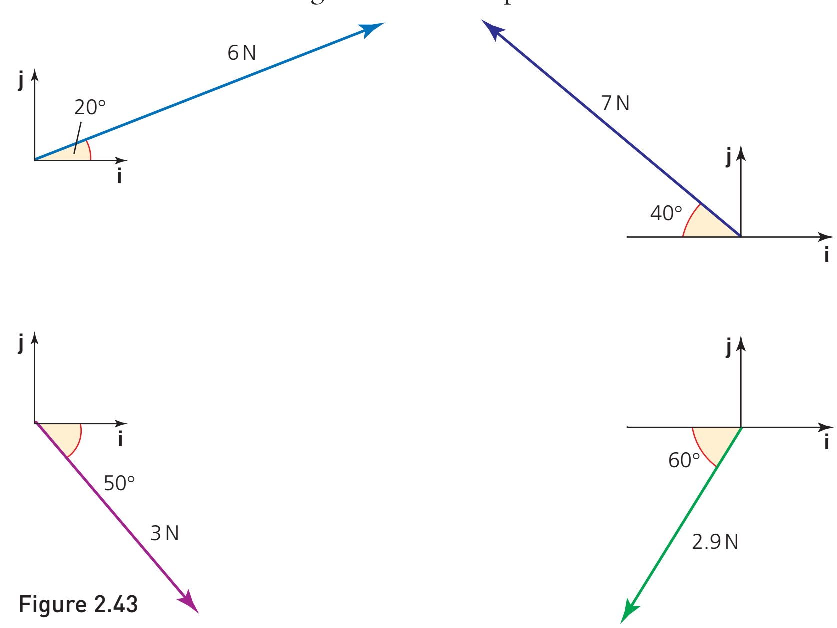


Figure 2.43

⑥ (i) Find a unit vector in the direction of $\begin{pmatrix} 10 \\ 24 \end{pmatrix}$.

A force **F** acts in the direction of $\begin{pmatrix} 10 \\ 24 \end{pmatrix}$ and has magnitude 39 N.

(ii) Use your answer to part (i) to write **F** in component form.

⑦ Find the vector with magnitude 8.2 that is parallel to the vector $40\mathbf{i} - 9\mathbf{j}$.

⑧ Write down each of the following vectors in terms of **i** and **j**. Find the resultant of each set of vectors in terms of **i** and **j**.

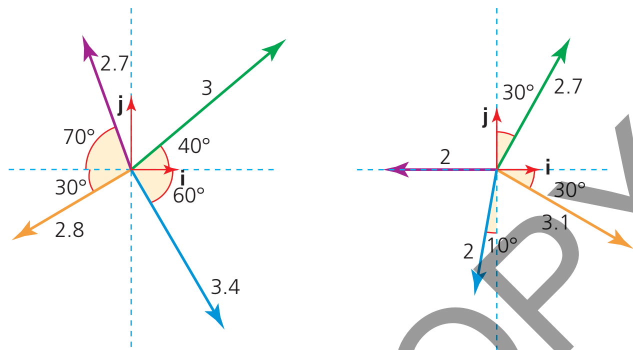


Figure 2.44

⑨ The displacement of B from A is $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$. The displacement of C from A is $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$. The displacement of D from A is $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$.

Draw a diagram showing the relative position of A, B, C and D. Find

(i) $\overrightarrow{DB}$ (ii) $\overrightarrow{DC}$ (iii) $\overrightarrow{CB}$ (iv) $\overrightarrow{BC}$

⑩ Three vectors **a**, **b** and **c** are represented by the sides of a triangle ABC, as shown in the diagram.

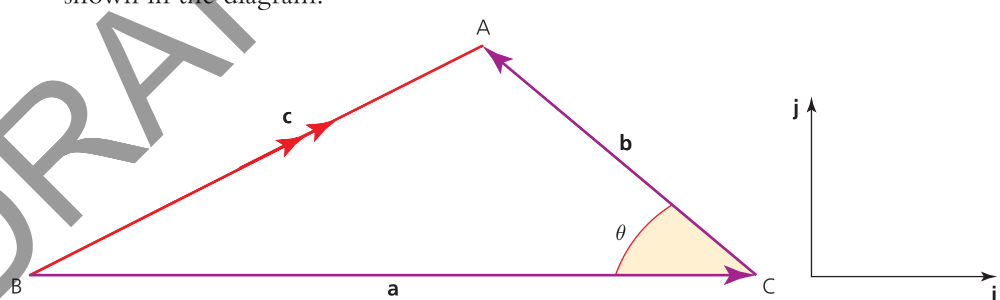


Figure 2.45

The angle C is θ and $|\mathbf{a}|$, $|\mathbf{b}|$ and $|\mathbf{c}|$ are a, b and c. Answer each part in terms of θ, a, b and c.

(i) Write **a** and **b** in terms of **i** and **j**.

(ii) Find $\mathbf{a} + \mathbf{b}$ and hence $|\mathbf{a} + \mathbf{b}|^2$.

(iii) Use your answer to part (ii) to express c^2 in terms of a, b and θ.

3 Forces in equilibrium

When forces are in equilibrium their vector sum is zero and the sum of the resolved parts in *any* direction is zero.

Example 2.10

A brick of mass 5 kg is at rest on a rough plane inclined at an angle of 35° to the horizontal. Find the frictional force F N, and the normal reaction R N of the plane on the brick.

Solution

The diagram shows the forces acting on the brick.

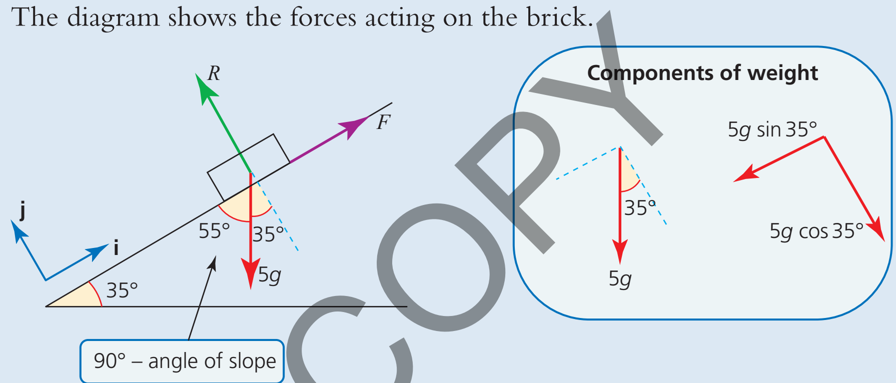


Figure 2.46

Take unit vectors **i** and **j** parallel and perpendicular to the plane, as shown.

Since the brick is in equilibrium, the resultant of the three forces acting on it is zero.

$5g = 49$

Resolving in the **i** direction: $F - 49 \sin 35° = 0$ ①

$F = 28.10 \ldots$

Resolving in the **j** direction: $R - 49 \cos 35° = 0$ ②

$R = 40.13 \ldots$

Written in vector form this is equivalent to

$$F\mathbf{i} + R\mathbf{j} - 49 \sin 35°\mathbf{i} - 49 \cos 35°\mathbf{j} = 0$$

or, alternatively,

$$\begin{pmatrix} F \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ R \end{pmatrix} + \begin{pmatrix} -49 \sin 35° \\ -49 \cos 35° \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Notice that both of these vector equations lead to the equations ① and ② above.

The triangle of forces

When there are only three (non-parallel) forces acting and they are in equilibrium, the polygon of forces becomes a closed triangle, as shown for the brick on the plane.

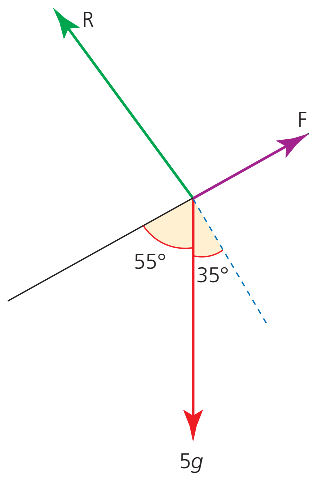


Figure 2.47

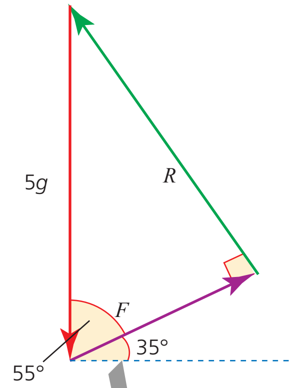


Figure 2.48

> **Note**
> The triangle is closed because the resultant is zero.

Then $$\frac{F}{5g} = \cos 55°$$

$$F = 49\cos 55° = 28.1\,\text{N}$$

And similarly $$R = 49\sin 55° = 40.1\,\text{N}$$

This is an example of the theorem known as the *triangle of forces.*

When a body is in equilibrium under the action of three non-parallel forces, then:

(i) the forces can be represented in magnitude and direction by the sides of a triangle

(ii) the lines of action of the forces pass through the same point. They are concurrent.

When more than three forces are in equilibrium, the first statement still holds but the triangle is then a polygon. The second statement is not necessarily true.

The next example illustrates two methods for solving problems involving forces in equilibrium. With experience, you will find it easier to judge which method is more suitable for a particular problem.

Example 2.11

A sign of mass 10 kg is to be suspended by two strings arranged as shown in the diagram below. Find the tension in each string.

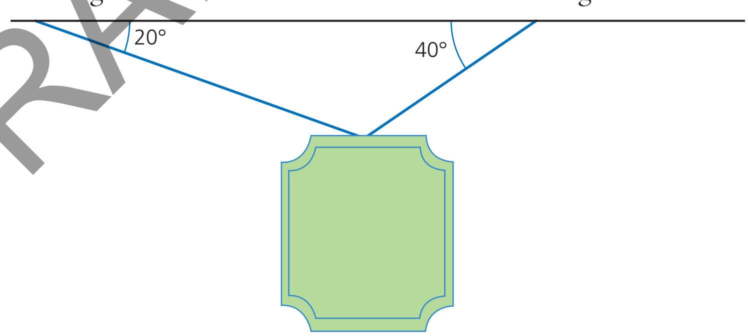


Figure 2.49

Solution

The force diagram for this situation is given below.

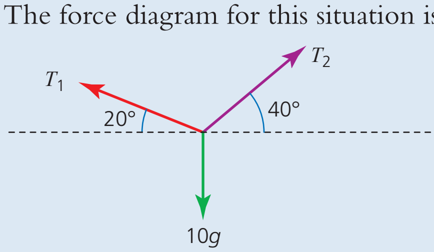


Figure 2.50

Method 1: Resolving forces

Vertically (↑): $T_1 \sin 20° + T_2 \sin 40° - 10g = 0$ ①

$$0.342\ldots T_1 + 0.642\ldots T_2 = 98$$

Horizontally (→): $-T_1 \cos 20° + T_2 \cos 40° = 0$ ②

$$-0.939\ldots T_1 + 0.766\ldots T_2 = 0$$

The set of simultaneous equations is solved in the usual way. Whether you are using the equation solver on your calculator or working it out on paper, it is important that you keep as much accuracy as possible by substituting for the different sines and cosines only at the very end of the calculation.

Multiply ① by $\cos 20°$ and then add ② $\times \sin 20°$ to give

$$T_2(\sin 40° \cos 20° + \cos 40° \sin 20°) = 98 \cos 20°$$

You may recognise that this is the compound angle form for $\sin(40° + 20°)$ and so is the same as $\sin 60°$.

$$T_2 = \frac{98 \cos 20°}{\sin 40° \cos 20° + \cos 40° \sin 20°}$$

$$T_2 = 106.33\ldots$$

Substituting back in ② now gives

$$T_1 = \frac{T_2 \cos 40°}{\cos 20°} = 86.68\ldots$$

The tensions in the strings are 86.7 N and 106.3 N.

Method 2: Triangle of forces

Since the three forces are in equilibrium they can be represented by the sides of a triangle taken in order.

Discussion point

In what order would you draw the three lines in this triangle of forces diagram?

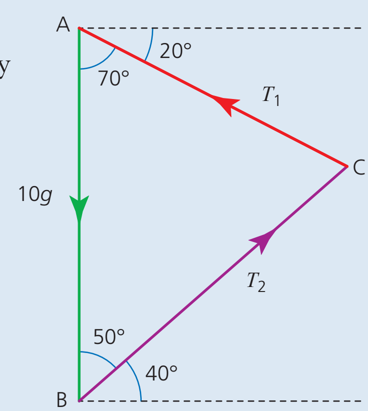


Figure 2.51

You can estimate the tensions by measurements. This will tell you that $T_1 \approx 87$ and $T_2 \approx 106$ in newtons.

Alternatively, you can use the sine rule to calculate T_1 and T_2 accurately.

In triangle ABC, $\widehat{CAB} = 70°$ and $\widehat{ABC} = 50°$, so $\widehat{BCA} = 60°$.

$180° - 70° - 50° = 60°$

So $$\frac{T_1}{\sin 50°} = \frac{T_2}{\sin 70°} = \frac{98}{\sin 60°}$$

giving $$T_1 = \frac{98 \sin 50°}{\sin 60°} \quad \text{and} \quad T_2 = \frac{98 \sin 70°}{\sin 60°}$$

As before, the tensions are found to be 86.7 N and 106.3 N.

Discussion point

Lami's theorem states that when three forces acting at a point, as shown in the diagram, are in equilibrium

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}.$$

Sketch a triangle of forces and say how the angles in the triangle are related to α, β and γ. Hence explain why Lami's theorem is true.

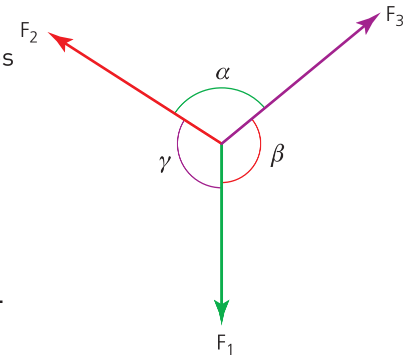


Figure 2.52

Example 2.12

Two husky dogs are pulling a sledge. They both exert forces of 60 N but at different angles to the line of the sledge, as shown in the diagram. The sledge is moving straight forwards.

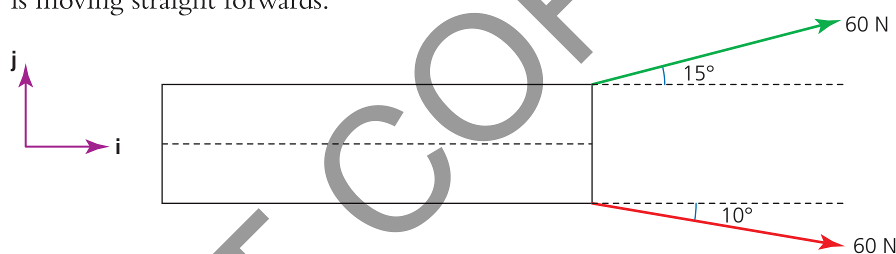


Figure 2.53

(i) Resolve the two forces into components parallel and perpendicular to the line of the sledge.

(ii) Find the overall forward force from the dogs and the overall sideways force.

The resistance to motion is 20 N along the line of the sledge and up to 400 N perpendicular to it.

(iii) Find the magnitude and direction of the overall horizontal force on the sledge.

(iv) How much force is lost due to the dogs not pulling straight forwards?

Solution

(i) Taking unit vectors **i** along the line of the sledge and **j** perpendicular to the line of the sledge.

The forces exerted by the two dogs are

$$60\cos 15°\,\mathbf{i} + 60\sin 15°\,\mathbf{j}$$

$$= 57.95\ldots\mathbf{i} + 15.52\ldots\mathbf{j}$$

and $$60\cos 10°\,\mathbf{i} - 60\sin 10°\,\mathbf{j}$$

$$= 59.08\ldots\mathbf{i} - 10.41\ldots\mathbf{j}$$

(ii) The overall forward force is equal to

$$60\cos 15° + 60\cos 10° = 117.04\ldots$$

The overall sideways force is equal to

$$60\sin 15° - 60\sin 10° = 5.11\ldots$$

(iii) The sideways force is cancelled by the resistance force opposing it.

The forward force is reduced by an amount 20 N from the resistance to motion.

So that the overall forward force is 97 N and the overall sideways force is 0.

The magnitude of the overall force on the sledge is thus 97 N in the direction of motion.

(iv) If the dogs were pulling straight, the overall force on the sledge would be 100 N, so the amount of force lost due to the dogs not pulling straight is thus 3 N.

60 + 60 – 20 (60 N from each dog less 20 N from the resistance)

100 – 97.04...

Exercise 2.3

① The following sets of forces are in equilibrium. Find the value of p and q in each case.

(i) $\begin{pmatrix}24\\18\end{pmatrix}$ N, $\begin{pmatrix}25\\60\end{pmatrix}$ N and $\begin{pmatrix}p\\q\end{pmatrix}$ N

(ii) $\begin{pmatrix}p\\-2\end{pmatrix}$ N, $\begin{pmatrix}-3\\4\end{pmatrix}$ N and $\begin{pmatrix}2\\-q\end{pmatrix}$ N

(iii) $\begin{pmatrix}2p\\5\end{pmatrix}$ N, $\begin{pmatrix}q\\4p\end{pmatrix}$ N, $\begin{pmatrix}p\\-3\end{pmatrix}$ N and $\begin{pmatrix}5\\-q\end{pmatrix}$ N.

② A brick of mass 2 kg is resting on a rough plane inclined at 40° to the horizontal.

(i) Draw a diagram showing all the forces acting on the brick.

(ii) Find the normal reaction of the plane on the brick.

(iii) Find the frictional force acting on the brick.

③ A particle is in equilibrium under the three forces shown in the diagram. Find the magnitude of the force F and the angle θ.

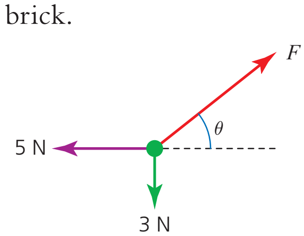


Figure 2.54

④ A box of mass 10 kg is at rest on a horizontal floor.

(i) Find the value of the normal reaction of the floor on the box.

The box remains at rest on the floor when a force of 30 N is applied to it at an angle of 25° to the upward vertical, as shown in the diagram.

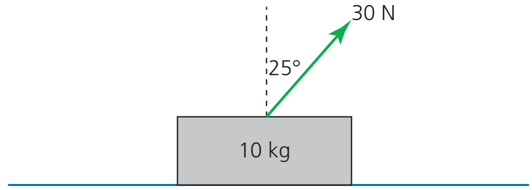


Figure 2.55

(i) Draw a diagram showing all the forces acting on the box.

(ii) Calculate the new value of the normal reaction of the floor on the box and also the frictional force.

⑤ A block of weight 100 N is on a rough plane that is inclined at 30° to the horizontal. The block is in equilibrium with a force of 35 N acting on it in the direction of the plane, as shown in the diagram.

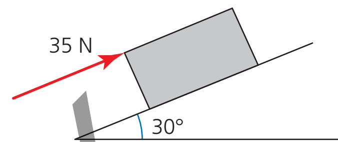


Figure 2.56

Calculate the frictional force acting on the block.

⑥ A crate of mass 10 kg is being pulled across rough horizontal ground by a rope making an angle θ with the horizontal. The tension in the rope is 60 N and the frictional force between the crate and the ground is 35 N. The forces on the crate are in equilibrium.

(i) Draw a labelled diagram showing all the forces acting on the crate.

(ii) Find the angle θ.

(iii) Find the normal reaction between the floor and the crate.

⑦ Each of three light strings has a block attached to one of its ends. The other ends of the strings are tied together at a point A. The strings are in equilibrium with two of them passing over fixed smooth pulleys and with the blocks hanging freely.

The weights of the blocks, and the angles between the sloping parts of the strings and the vertical, are as shown in the diagram. Find the values of W_1 and W_2.

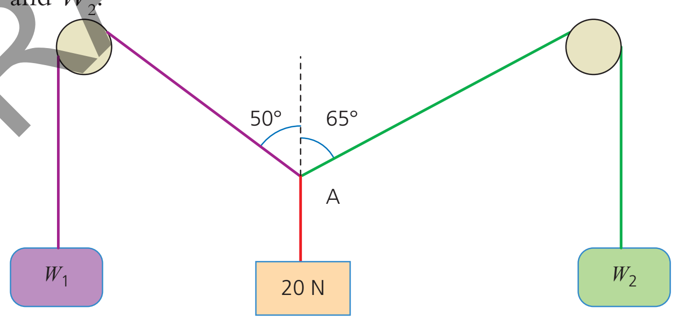


Figure 2.57

⑧ A block of mass 10 kg rests in equilibrium on a smooth plane inclined at 30° to the horizontal. It is held by a light string making an angle of 15° with the line of greatest slope of the plane.

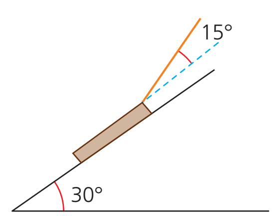


Figure 2.58

(i) Draw a labelled diagram showing all the forces acting on the block.

(ii) Find the tension in the string and the normal reaction of the plane on the block.

9 A particle A of mass 3 kg is at rest in equilibrium on horizontal rough ground. A is attached to two light, inextensible strings making angles of 20° and 50° with the vertical. The tensions in the two strings are 10 N and 20 N, as shown in the diagram.

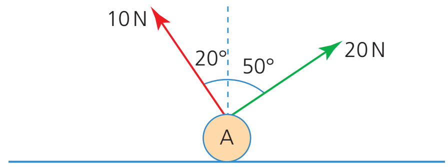


Figure 2.59

(i) Draw a diagram showing all the forces acting on A.

(ii) Find the normal reaction between the ground and A.

(iii) Find the magnitude of the frictional force, indicating the direction in which it is acting.

10 Four wires, all of them horizontal, are attached to the top of a telegraph pole as shown in this plan view. There is no resultant force on the pole and tensions in the wires are as shown.

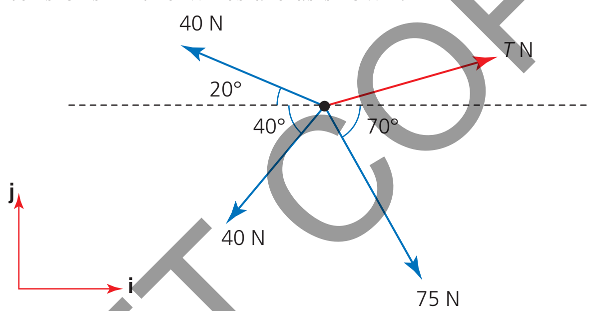


Figure 2.60

(i) Using perpendicular directions as shown in the diagram, show that the force of 75 N may be written as $(25.7\mathbf{i} - 70.5\mathbf{j})$ N (to 3 significant figures).

(ii) Find T in both component form and magnitude and direction form.

(iii) The force T is changed to $(40\mathbf{i} + 50\mathbf{j})$ N. Show that there is now a resultant force on the pole and find its magnitude and direction.

11 A ship is being towed by two tugs. They exert forces on the ship as indicated.

There is also a drag force on the ship.

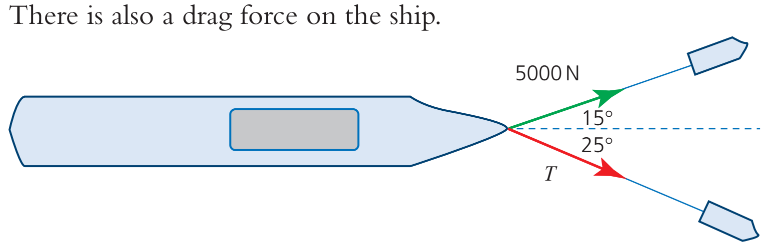


Figure 2.61

(i) Write down the components of the tensions in the towing cables along and perpendicular to the line of motion, l, of the ship.

(ii) Show that there is no resultant force perpendicular to the line l. Find T.

(iii) The ship is travelling with constant velocity along the line l. Find the magnitude of the drag force acting on it.

⑫ A skier of mass 60 kg is skiing down a slope inclined at 20° to the horizontal.

(i) Draw a diagram showing the forces acting on the skier.

(ii) Resolve these forces into components parallel and perpendicular to the slope.

(iii) The skier is travelling at constant speed. Find the normal reaction of the slope on the skier and the resistive force on her.

(iv) The skier later returns to the top of the slope by being pulled up it at constant speed by a rope parallel to the slope. Assuming the resistance on the skier is the same as before, calculate the tension in the rope.

⑬ The diagram shows a block of mass 10 kg on a rough inclined plane. The block is attached to a 7 kg weight by a light string which passes over a smooth pulley; it is on the point of sliding up the slope.

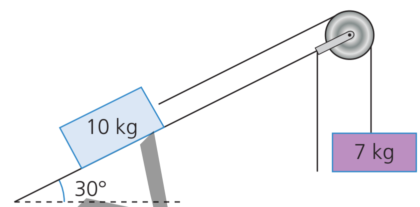


Figure 2.62

(i) Draw a diagram showing the forces acting on the block.

(ii) Resolve these forces into components parallel and perpendicular to the slope.

(iii) Find the force of resistance to the block's motion.

The 7 kg mass is replaced by one of mass m kg.

(iv) Find the value of m for which the block is on the point of sliding down the slope, assuming the resistance to motion is the same as before.

⑭ The diagram shows a sign attached to a point A by a light rigid rod AB. It is supported by two light rigid rods AC and AD. AC is horizontal and AD makes an angle θ with the horizontal with $\tan\theta = 0.75$. The mass of the sign is 20 kg.

Find the forces in the rods AB, AC and AD, stating whether they are in tension or compression.

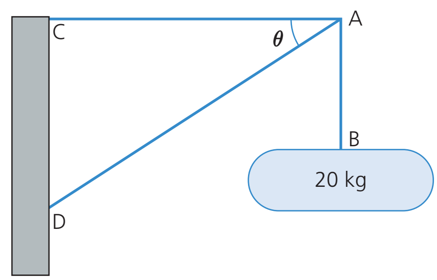


Figure 2.63

⑮ A block of mass 75 kg is in equilibrium on smooth horizontal ground with one end of a light string attached to its upper edge. The string passes over a smooth pulley, with a block of mass m kg attached at the other end.

The part of the string between the pulley and the block makes an angle of 65° with the horizontal. A horizontal force F is also acting on the block.

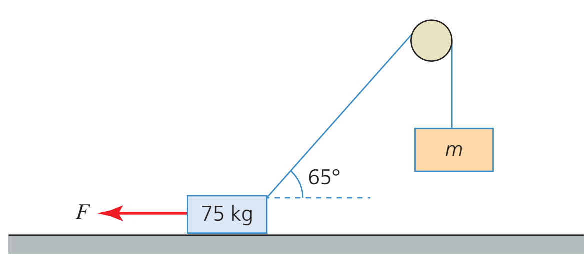


Figure 2.64

(i) Find a relationship between T, the tension in the string, and R, the normal reaction between the block and the ground.

The block is on the point of lifting off the ground.

(ii) Find T and m.

(iii) Find F.

⑯ Two boxes of masses 15 kg and 12 kg are held by light strings AB, BC and CD. As shown in the figure, AB makes an angle α with the horizontal and is fixed at A.

Angle α is such that $\sin\alpha = 0.28$. BC is horizontal and CD makes an angle β with the horizontal.

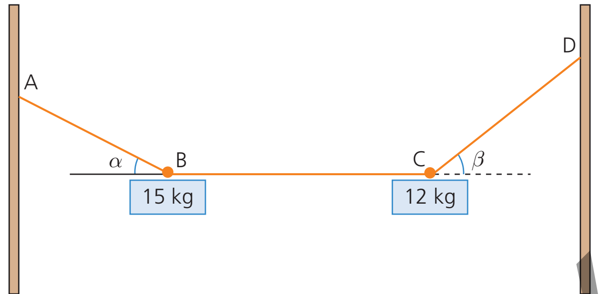


Figure 2.65

(i) By considering the equilibrium of point B, find the tension in string AB and show that the tension in string BC is 504 N.

(ii) Find β and the tension in CD.

4 Finding resultant forces

When forces are in equilibrium, their resultant is zero. However, forces are not always in equilibrium. The next example shows you how to find the resultant of forces that are not in equilibrium. You know from Newton's second law that the acceleration of the body will be in the same direction as the resultant force; remember that force and acceleration are both vector quantities.

Example 2.13

A sledge is being pulled up a smooth slope inclined at an angle of 15° to the horizontal by a rope which makes an angle of 30° with the slope. The mass of the sledge is 5 kg and the tension in the rope is 40 N.

(i) Draw a diagram to show the forces acting on the sledge.

(ii) Find the resultant of these forces.

(iii) Find the acceleration of the sledge.

Note

When the sledge is modelled as a particle, all the forces can be assumed to be acting at a point.

Solution

(i) Figure 2.66 is the force diagram.

(ii) ***Method 1***

Resolve the forces into components parallel and perpendicular to the slope.

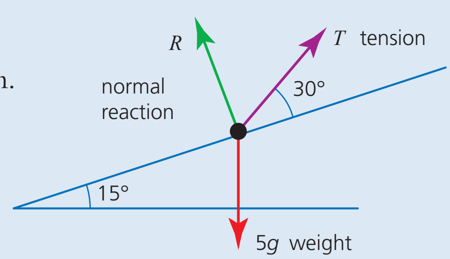


Figure 2.66

Note

There is no frictional force because the slope is smooth.

Hint

Notice that although the sledge is moving up the slope this does not mean that the resultant force is up the slope. Its direction depends on the acceleration of the sledge which may be up or down the slope, or zero if the sledge is moving at constant speed.

Discussion point

Try resolving horizontally and vertically. You will obtain 2 equations in the two unknowns F and R. It is perfectly possible to solve these equations, but is quite a lot of work. How can you decide which directions will be easiest to work with?

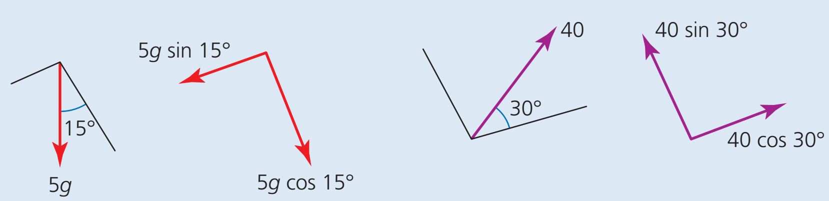


Figure 2.67

Resolve parallel to the slope: →

The resultant is $F = 40\cos 30° - 5g\sin 15°$
$= 21.959\ldots$

The force $\mathbf{R}$ is perpendicular to the slope so it has no component in this direction

Resolve perpendicular to the slope: →

$$R + 40\sin 30° - 5g\cos 15° = 0$$
$$R = 5g\cos 15° - 40\sin 30° = 27.33$$

There is no resultant force in this direction because the motion is parallel to the slope

To 3 significant figures, the normal reaction is 27.3 N and the resultant force is 22.0 N up the slope.

Method 2

Alternatively, you could have worked in column vectors as follows.

$$\begin{pmatrix} 0 \\ R \end{pmatrix} + \begin{pmatrix} 40\cos 30° \\ 40\sin 30° \end{pmatrix} + \begin{pmatrix} -5g\sin 15° \\ -5g\cos 15° \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

Parallel to slope

Perpendicular to slope

normal reaction + tension + weight = resultant force

(iii) Once you know the resultant force you can work out the acceleration of the sledge using Newton's second law.

$$F = ma$$
$$21.959\ldots = 5a$$
$$a = \frac{21.959\ldots}{5} = 4.392\ldots$$

The acceleration is $4.4\,\mathrm{m\,s^{-2}}$ (correct to 1 d.p.)

The resultant force is in the direction of motion and so must be parallel to the slope

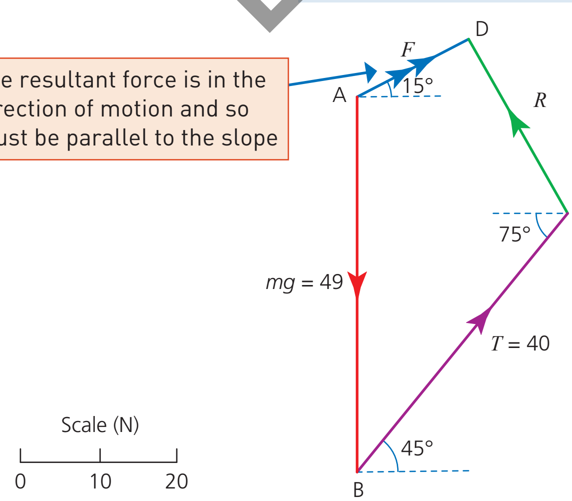


Figure 2.68

An alternative way of approaching the previous example is to draw a scale diagram with the three forces represented by three of the sides of a quadrilateral taken in order (with the arrows following each other, $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CD}$), as shown in Figure 2.68. The resultant is represented by the fourth side AD.

From the diagram, you can estimate the normal reaction to be about 30 N and the resultant 20 N.

Discussion point

In what order would you draw the lines in the diagram?

Discussion point

What can you say about the acceleration of the sledge in the previous example, in the cases when:

(i) the length AD in Figure 2.68 on the previous page is not zero?

(ii) the length AD is zero so that the starting point on the quadrilateral is the same as the finishing point?

(iii) BC is so short that the point D is to the left of A, as shown in Figure 2.69?

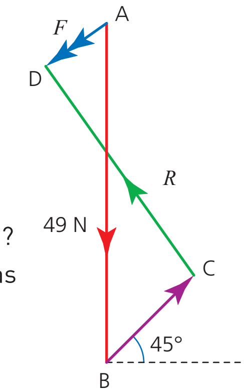


Figure 2.69

Example 2.14

Two forces **P** and **Q** act at a point O on a particle of mass 2 kg. Force **P** has magnitude 50 N and acts along a bearing of 030°. Force **Q** has magnitude of 30 N and acts along a bearing of 315°.

(i) Find the magnitude and bearing of the resultant force **P** + **Q**.

(ii) Find the acceleration of the particle.

Solution

(i) Forces **P** and **Q** are illustrated below.

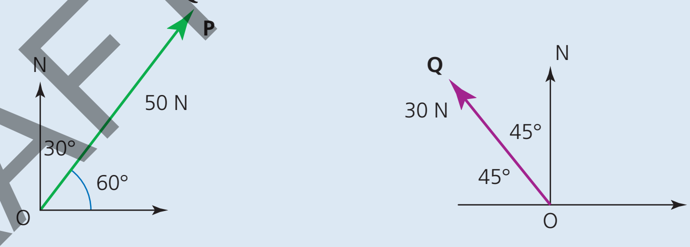


Figure 2.70

Note

Notice that **P** and are written as vectors.

$$\mathbf{P} = \begin{pmatrix} 50\cos 60° \\ 50\sin 60° \end{pmatrix}$$

$$= \begin{pmatrix} 25 \\ 43.30\ldots \end{pmatrix}$$

$$\mathbf{Q} = \begin{pmatrix} -30\cos 45° \\ 30\sin 45° \end{pmatrix}$$

$$= \begin{pmatrix} -21.21\ldots \\ 21.21\ldots \end{pmatrix}$$

$$\mathbf{P}+\mathbf{Q} = \begin{pmatrix} 25 \\ 43.30\ldots \end{pmatrix} + \begin{pmatrix} -21.21\ldots \\ 21.21\ldots \end{pmatrix}$$

$$= \begin{pmatrix} 3.78\ldots \\ 64.51\ldots \end{pmatrix}$$

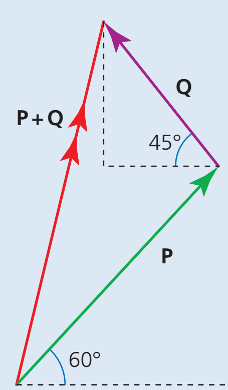


Figure 2.71

The resultant is shown in Figure 2.72.

Magnitude $|\mathbf{P}+\mathbf{Q}| = \sqrt{3.78...^2 + 64.51...^2}$

$= 64.62$

Direction $\tan\theta = \dfrac{64.51...}{3.78...}$

$\theta = 86.64...°$

The bearing is $90° - 86.64...° = 3.36...°$

The force $\mathbf{P} + \mathbf{Q}$ has magnitude 65 N and bearing 003°.

(ii) The acceleration of the particle is given by

$$\mathbf{a} = \frac{1}{2}(\mathbf{P}+\mathbf{Q}) = \frac{1}{2}\begin{pmatrix} 3.78\ldots \\ 64.51\ldots \end{pmatrix} = \begin{pmatrix} 1.89 \\ 32.3 \end{pmatrix}$$

$m = 2\,\text{kg}$

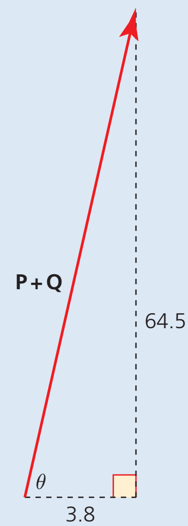


Figure 2.72

The magnitude of the acceleration is $= \sqrt{1.89^2 + 32.3^2} = 32.4$.

The bearing is $\arctan\left(\dfrac{1.89}{32.3}\right) = 3.35°$.

$\mathbf{P}$ and $\mathbf{Q}$ give the particle an acceleration of $32.3\,\text{m}\,\text{s}^{-2}$ on a bearing of 003°.

Sometimes, as in the next example, it is just as easy to work with the trigonometry of the diagram as with the components of the forces.

Example 2.15

The angle between the lines of action of two forces $\mathbf{X}$ and $\mathbf{Y}$ is θ. Find the magnitude and direction of the resultant.

Solution

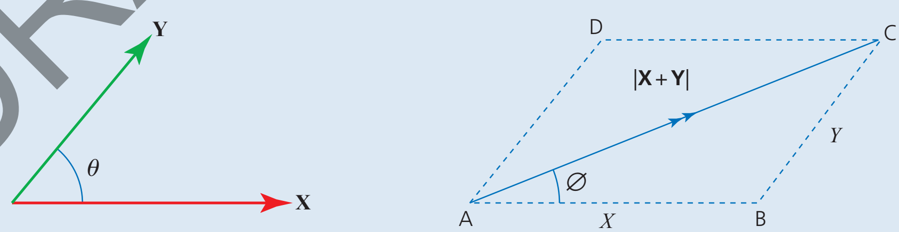


Figure 2.73

Use the cosine rule in triangle ABC. The magnitude of the resultant is F.

$$F = |\mathbf{X}+\mathbf{Y}| = \sqrt{\text{AB}^2 + \text{BC}^2 - 2\text{AB}\times\text{BC}\times\cos(\widehat{\text{ABC}})}$$

$$= \sqrt{X^2 + Y^2 - 2XY\cos(180° - \theta)}$$

$$= \sqrt{X^2 + Y^2 + 2XY\cos\theta}$$

Use the sine rule in triangle ABC. The resultant makes an angle ϕ with the $\mathbf{X}$ force.

$$\frac{\sin \widehat{CAB}}{BC} = \frac{\sin \widehat{ABC}}{AC}$$

$$\frac{\sin \varnothing}{Y} = \frac{\sin(180^\circ - \theta)}{F}$$

$$\sin \varnothing = \frac{Y}{F} \sin \theta$$

$$\varnothing = \arcsin\left(\frac{Y}{F} \sin \theta\right)$$

The resultant of the two forces **X** and **Y** inclined at θ has magnitude $F = \sqrt{X^2 + Y^2 + 2XY \cos\, \theta}$ and makes an angle $\arcsin\left(\frac{Y}{F} \sin \theta\right)$ with the **X** force.

Exercise 2.4

For questions 1–6, carry out the following steps. All forces are in newtons.

(i) Draw a scale diagram to show the forces and their resultant.

(ii) State whether you think the forces are in equilibrium and, if not, estimate the magnitude and direction of the resultant.

(iii) Write the forces in component form, using the directions indicated and so obtain the components of the resultant. Hence find the magnitude and direction of the resultant.

(iv) Compare your answers to parts (ii) and (iii).

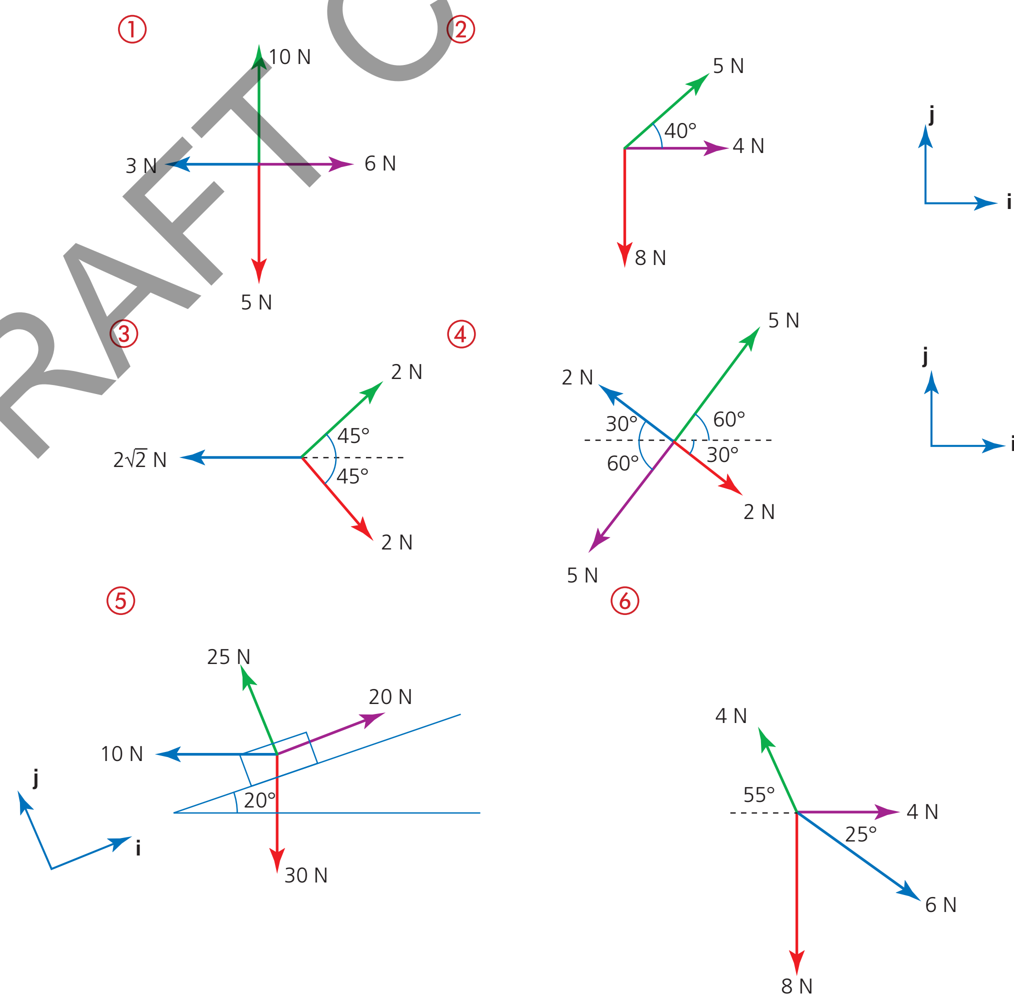


Figure 2.74

7. Four horizontal wires are attached to a telephone post and exert the following tensions on it: 25 N in the north direction, 30 N in the east direction, 45 N in the north-west direction and 50 N in the south-west direction. Calculate the resultant tension on the post and find its direction.

8. Forces of magnitude 7 N, 10 N and 15 N act on a particle of mass 1.5 kg in the directions shown in the diagram.

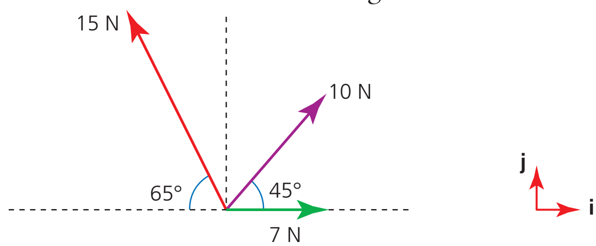


Figure 2.75

(i) Find the components of the resultant of the three forces in the **i** and **j** directions.

(ii) Find the magnitude and direction of the resultant.

(iii) Find the acceleration of the particle.

9. (i) Find the resultant of the set of 6 forces whose magnitudes and directions are shown in the diagram below.

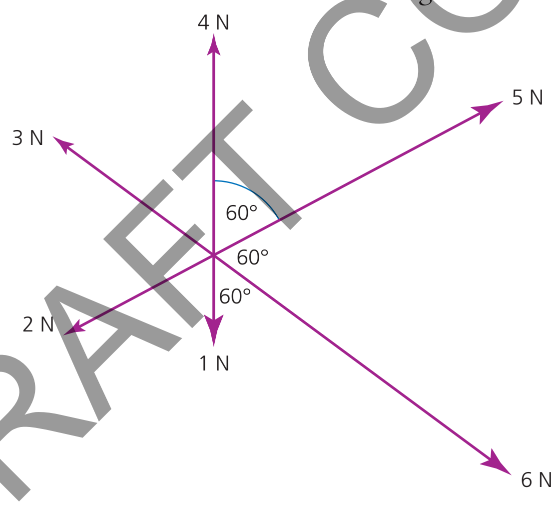


Figure 2.76

The forces are acting on a particle P of mass 5 kg which is initially at rest at O.

(ii) How fast is P moving after 3 s and how far from O is it now?

10. A force **P** of magnitude 5 N makes an angle of 60° with a force **Q** of magnitude 4 N. Find the magnitude of the resultant force and the angle it makes with the **P** force.

11. Two forces **P** and **Q** are at right angles to each other. The resultant has magnitude 20 N and makes an angle 60° with **P**. Find the magnitude of **P** and **Q**.

12. The resultant of two forces **P** and **Q** acting on a particle has magnitude $P = |\mathbf{P}|$. The resultant of the two forces $3\mathbf{P}$ and $2\mathbf{Q}$ acting in the same directions as before has magnitude $2P$. Find the magnitude of **Q** and the angle between **P** and **Q**.

KEY POINTS

1 Newton's laws of motion

- Every object continues in a state of rest or uniform motion in a straight line unless it is acted on by an external force.
- Resultant force = mass × acceleration or $\mathbf{F} = m\mathbf{a}$
- When one object exerts a force on another there is always a reaction force which is equal and opposite in direction to the acting force.

2 Force is a vector. It may be represented in either magnitude–direction form or in component form

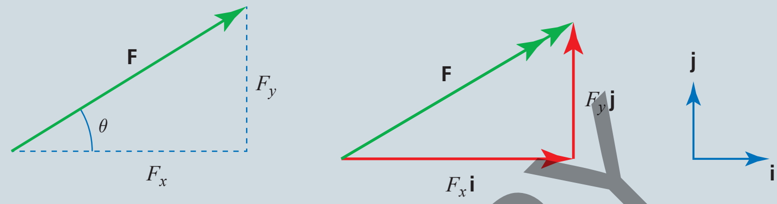


Figure 2.77

Magnitude of $\mathbf{F} = |\mathbf{F}| = \sqrt{F_x^{\,2} + F_y^{\,2}}$ $\quad \mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}$

Direction of $\mathbf{F}$ $\quad \theta = \arctan\left(\dfrac{F_y}{F_x}\right)$

3 Resolving forces

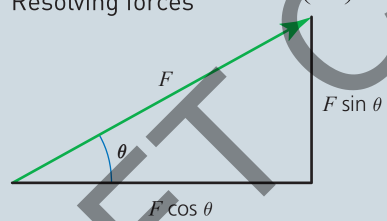


Figure 2.78

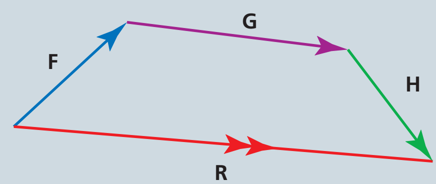


Figure 2.79

$$\mathbf{F} = F\cos\theta\mathbf{i} + F\sin\theta\mathbf{j} = \begin{pmatrix} F\cos\theta \\ F\sin\theta \end{pmatrix}$$

4 Resultant forces

$$\mathbf{R} = \mathbf{F} + \mathbf{G} + \mathbf{H} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} + \begin{pmatrix} G_x \\ G_y \end{pmatrix} + \begin{pmatrix} H_x \\ H_y \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} X \\ Y \end{pmatrix},\ X = F_x + G_x + H_x,\ Y = F_y + G_y + H_y$$

Magnitude of $\mathbf{R} = \sqrt{X^2 + Y^2}$ $\quad$ Direction of $\mathbf{R}$ $\theta = \arctan\left(\dfrac{Y}{X}\right)$

5 Equilibrium

When the resultant is zero, the forces are in equilibrium.

6 Triangle of forces

If an object is in equilibrium under three non-parallel forces, their lines of action are concurrent and they can be represented by a triangle.

LEARNING OUTCOMES

When you have finished this chapter, you should be able to

- draw a diagram showing the forces acting on a body
- apply Newton's laws of motion to problems in one or more dimensions
- resolve a force into components having selected suitable directions for resolution
- find the resultant of several concurrent forces
- realise that a particle is in equilibrium under a set of concurrent forces if and only if the resultant force is zero
- know that a closed polygon may be drawn to represent the forces acting on a particle in equilibrium
- formulate equations for equilibrium by resolving forces in suitable directions
- formulate the equation of motion of a particle which is being acted on by several forces
- know that contact between two surfaces is lost when the normal reaction force becomes zero.